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Exact closed-form solution for the vibration modes of the Euler–Bernoulli beam with multiple open cracks

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ABSTRACT

In this study, exact closed-form expressions for the vibration modes of the Euler–Bernoulli beam in the presence of multiple concentrated cracks are presented. The proposed expressions are provided explicitly as functions of four integration constants only, to be determined by the standard boundary conditions. The enforcement of the boundary conditions leads to explicit expressions of the natural frequency equations. Besides the evaluation of the natural frequencies, neither computational work nor recurrence expressions for the vibration modes are required.

The cracks, that are not subjected to the closing phenomenon, are modelled as a sequence of Dirac's delta generalised functions in the flexural stiffness. The Eigen-mode governing equation is formulated over the entire domain of the beam without enforcement of any continuity conditions, which are already accounted for in the adopted flexural stiffness model. The vibration modes of beams with different numbers of cracks under different boundary conditions have been analysed by means of the proposed closed-form expressions in order to show their efficiency.

1. Introduction

Several studies have been conducted in the last decade aimed at the detection and identification of damages in civil and mechanical structures, in order to assess their general health or their residual load carrying capacity. Health monitoring and damage detection can be conducted by executing non-destructive vibration tests in view of the easy repeatability of such tests. Damage identification on the basis of dynamic measurements is usually conducted by means of numerical procedures in view of the difficulty in obtaining explicit solutions of both the direct and inverse analysis problem. For the latter reason, the study of simple structural systems, such as straight beams subject to concentrated damages, has been conducted by several authors [1–20]. A few authors have successfully addressed aspects regarding the identification of single cracks, leading to explicit expressions for the solution of the inverse problem [5]; while the assessment of the variation of the natural frequencies on account of damage and position of a concentrated crack has been clarified in [4,6,11]. However, several characteristics, particularly concerning the case of identification of damage and positions of multiple cracks by means of dynamic tests, remain unsolved.

In recent years, greater attention has been devoted to the solution of the direct analysis problem of vibrating beams in the presence of multiple concentrated cracks [20–26]. In fact, a deeper insight and understanding of the latter problem is fundamental to addressing the well-posed inverse formulation.

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Several studies introduced concentrated damages on beams by considering a local reduction of the flexural stiffness. According to the latter model, and following the examples in [27–30], a crack can be macroscopically represented as an elastic link connecting the two adjacent beam segments. More precisely, a model in which an internal hinge endowed with a rotational spring, whose stiffness is dependent on the extent of damage, proved to be accurate and is often used [31–36].

The effect of cracks on the dynamic properties of the beam has been mainly studied with particular attention given to the characteristic equation of the natural frequencies. The first studies were based on the sub-division of the beam into sub-beams between two subsequent cracks, requiring the enforcement of four continuity conditions at each cracked cross-section [34,35]. According to this procedure, the characteristic equation of a beam with n cracks relies on the solution of a determinant of order $4(n+1)$. More recently, Shifrin and Ruotolo [21] proposed the smooth function method, which reduced the determinant to order $n+1$. However, from a computational viewpoint, the number of cracks is always influent. The classical method, consisting of the beam sub-division, is also usually adopted to treat the case of vibration of stepped beams showing abrupt variations of the cross-section as, for example, in [37]. However, the procedure proposed in [37] is not competitive in the case of multiple discontinuities, particularly if closed form solutions are sought.

The approach proposed by Khiem and Lien [22] is based on the transfer matrix procedure, and can relate the state variables at the end of each sub-beam with the values at the first end. In the latter method, the fulfilling of continuity conditions at the cracked cross-sections allowed the authors to relate the state variables at both ends of the entire beam; therefore, the characteristic equation of the natural frequencies is found by solving a determinant of order four. A thorough parametric study on the influences of the intensity, position, and number of the cracks on the natural frequency was conducted in [22].

Ruotolo and Surace [24] extended the smooth function method proposed in [21] and the transfer matrix method, proposed by Khiem and Lien for bending vibrations [22], to the evaluation of the longitudinal natural frequencies of vibrating bars with an arbitrary number of cracks. The transfer matrix method was also adopted successfully by Sorrentino et al. [38] to treat the case of stepped Timoshenko beams, and in particular, to address the case of generalised damping coefficients implying non-proportional damping.

An appealing approach is that proposed by Li [23], who utilised a properly derived basic solution exploiting the fundamental solutions for each sub-beam and obtained the equation of the natural frequencies by solving a second-order determinant. The expressions of the mode shapes are provided for each sub-beam, and since they are dependent on the preceding values of the mode shapes, are given by means of recurrence formulas.

An extension of the procedure proposed by Li [23] to the case of beams with multiple cracks in the presence of an axial force was proposed by Binici [25]. Once again, by selecting the appropriate fundamental solutions, the Eigen-value equation can be conveniently determined by evaluating a second-order determinant, while the mode shapes are provided by recurrence formulas in terms of the initial parameters that satisfy the boundary conditions.

The use of generalised functions (distributions) proved to be an efficient tool to treat discontinuous functions. Shifrin and Ruotolo [21] accounted for the discontinuities in the displacement function and its derivatives by making use of distributions. Yavari and co-workers [37,38] studied the static response of beams with multiple discontinuities, and used the theory of distributions to show that a unique displacement function can be used to describe the behaviour of the entire beam in the presence of discontinuities. A wide variety of cases involving discontinuities of the external loads, the cross-section, the transversal displacements, and the rotation function have been studied with reference to both the Euler–Bernoulli and the Timoshenko beams [39,40]. The procedure proposed in these papers to treat the governing equations within the context of distributions has also been adopted for the case of columns in the presence of discontinuities [41] and the case of Euler–Bernoulli and Timoshenko beams with jump discontinuities on discontinuous foundations in a static context [42]. However, their solution procedure requires the enforcement of n continuity conditions, and no explicit expressions of the solution are provided.

The structural problem studied by Yavari and Sarkani [41], i.e. the governing equation of a beam with jump discontinuous flexural stiffness, was addressed by Horman and Oparnica in [43]. In this work, in accordance with [41], it is proved that a differential equation governing the transversal displacement function of beams with jump discontinuous coefficients, cannot possess a "distributional solution" in the context of the distributional product hierarchy (described by the same authors in their appendix), if the solution displacement function shows a jump discontinuity at the same cross-section. However, this work considers neither jump discontinuities of the flexural stiffness nor jump discontinuities of the transversal displacements. Stankovic and Atanakovic [44] studied linear discontinuous differential equations, and provided weak solutions to some mathematical models in mechanics [39–42] where discontinuities appear.

Very recently, Wang and Qiao [26] examined the case of vibrations of beams in the presence of any type of discontinuities. They expressed the modal displacement function of the entire beam in the presence of n discontinuities by means of a single function through the use of distributions. For each discontinuity, at a given location, a basic modal displacement function starting at that point is introduced. The modal displacement function of the entire beam is obtained as the superposition of these basic functions, and is given under the form of a recurrence expression; the equation of the natural frequencies can be obtained by solving a second-order determinant. The recurrence expressions proposed by Li [23] can be recovered as special cases of those proposed by Wang and Qiao [26].

In view of the above-mentioned literature, to the authors' knowledge no explicit closed form solution for the vibration mode shapes is available. On the contrary, the best results proposed in the literature are based on recurrence expressions, which require knowledge of the solution at the prior cracked cross-sections to evaluate the mode shape at specific cross-sections [23,26].

The aim of the present work is to provide exact closed-form expressions for the mode shapes of the vibrating Euler–Bernoulli beam with multiple concentrated cracks.

Within the framework of the distribution theory, a model for columns in the presence of multiple concentrated cracks was presented in [45]. The authors proposed an integration procedure of the governing equation to obtain explicit solutions of the critical load equation and of the corresponding buckling modes.

In this study, an extension of this procedure to treat the case of damaged vibrating beams in the presence of multiple open concentrated cracks is studied. The cracks are modelled by means of the introduction of a sequence of Dirac's delta generalised functions in the flexural stiffness, which is equivalent to a sequence of concentrated elastic rotational springs [45–48]. The governing equation for the proposed model is a differential equation with discontinuous coefficients. However, the structure of the governing equation does not fall into the categories studied in [43,44].

A procedure to describe vibrations of the Euler–Bernoulli beam in the presence of discontinuities, including the case of stepped beams, which relies on the knowledge of the closed-form expressions of the static Green's functions, was proposed by Failla and Santini [49]. The latter expressions are reported in the appendix of their paper and are directly obtained from the closed-form solutions provided in [47]. However the dynamic solution procedure proposed in [49] leads to approximate solutions for stepped beams, since a lumped-mass discretisation is used.

In the present work, a suitable procedure for the exact integration of the governing equation of the mode shapes in the presence of singularities in the flexural stiffness, representing multiple concentrated cracks, is proposed. Explicit exact closed-form solutions for the mode shapes, together with the explicit expressions of the natural frequency equations, are obtained. The explicit expressions of the mode shapes are given as functions of four integration constants solely dependent on the boundary conditions, and show the classical analytical structure of those concerning the undamaged beams.

The above-mentioned solution proposed for the vibration modes is of great help to researchers working towards the evaluation of the dynamic response of structures composed of damaged beams through exact modal analysis by making use of the dynamic stiffness matrix. Moreover, the results proposed in the manuscript can also be employed by those authors dealing with nonlinear responses due to the phenomenon of closing cracks by means of a sequence of linear analyses (piecewise linear analysis). Furthermore, the knowledge of the exact solution of the direct problem can also be used to tackle inverse problems such as damage identification based on experimental measurements of the Eigen-modes of beams subject to concentrated cracks.

A parametric analysis has been conducted for different boundary conditions in order to investigate the influence of the number, position, and intensity of the cracks on the dynamical properties of the Euler–Bernoulli beam.

2. The vibration modes of the Euler–Bernoulli beam with multiple concentrated cracks

The basic concept adopted in this study is that concentrated cracks locally affect the flexural stiffness of the beam, and that their influence can be modelled via generalised functions. More precisely, the Euler–Bernoulli beam in the presence of a single crack concentrated at abscissa x_0 along the axis can be modelled [46] using Dirac's delta centred at x_0 in the flexural stiffness $E(x)I(x)$ of the beam (the Young modulus $E(x)$ multiplied by the moment of inertia $I(x)$ of the cross-section) as follows:

$$E(x)I(x) = E_0 I_0[1 - \hat{\gamma}\delta(x - x_0)], \tag{1}$$

where E_0I_0 is the uniform flexural stiffness of the beam and $\hat{\gamma}$ is a parameter representative of the crack intensity. The equivalence between the model proposed in Eq. (1) and the standard differential equations of a beam subject to a single crack is shown in Appendix A, where the relationship between the parameter $\hat{\gamma}$ and the crack intensity is also specified.

In this section, the Euler–Bernoulli vibrating beam subject to multiple concentrated damages is considered by making use of the model presented in Eq. (1). Specifically, an integration procedure able to provide the closed form expression of the mode shapes is presented. According to the proposed model, the following expression of uniform flexural stiffness with Dirac's delta singularities is adopted in this study to treat an arbitrary number of concentrated cracks:

$$E(x)I(x) = E_0 I_0 \left[1 - \sum_{i=1}^{n} \hat{\gamma}_i \delta(x - x_{0i})\right], \tag{2}$$

where n singularities, given by Dirac's deltas centred at abscissa x_{0i}, $i = 1,\dots,n$, represent n concentrated cracks. The parameters $\hat{\gamma}_i$, $i = 1,\dots,n$ introduced in Eq. (2) multiplying the Dirac's deltas are related to the rotational stiffness of an equivalent internal spring, as specified in Appendix A.

In view of Eq. (2), the differential equation governing the free vibration of a beam with multiple concentrated cracks may be given the following form:

$$\left[E_0 I_0 \left(1 - \sum_{i=1}^{n} \hat{\gamma}_i \delta(x - x_{0i})\right) \overline{u''}(x,t)\right]'' + m\ddot{u}(x,t) = 0, \tag{3}$$

where the apex indicates the standard derivative with respect to the spatial variable x, the superimposed bar indicates the distributional derivative, and the superimposed dot indicates the standard derivative with respect to time t.

By introducing a non-dimensional coordinate $\xi = x/L$, the flexural stiffness model introduced in Eq. (2) can be modified as follows:

$$E(\xi)I(\xi) = E_0 I_0 \left[1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right]. \tag{4}$$

Therefore, the governing differential equation (3) can be written as

$$\left[\left(1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right)\overline{u''}(\xi, t)\right]'' + \frac{mL^4}{E_0 I_0}\ddot{u}(\xi, t) = 0. \tag{5}$$

In Eq. (4) the property $\delta(x - x_{0i}) = \delta[L(\xi - \xi_{0i})] = (1/L)\delta(\xi - \xi_{0i})$ of Dirac's delta distribution has been utilised [50–55], and the dimensionless parameters $\gamma_i = \hat{\gamma}_i/L$ is introduced.

The displacement $u(\xi, t)$ can be assumed as the product of the function $\phi(\xi)$, which depends on the spatial coordinate ξ and a time dependent function $y(t)$, as follows:

$$u(\xi, t) = y(t)\phi(\xi). \tag{6}$$

Substitution of the separated form provided by Eq. (6) into the governing equation (5) yields the following differential equation:

$$\left[\left(1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right)\overline{\phi''}(\xi)\right]'' y(t) + \frac{mL^4}{E_0 I_0}\phi(\xi)\ddot{y}(t) = 0. \tag{7}$$

Eq. (7), after simple manipulations, can be written as

$$\frac{E_0 I_0}{mL^4}\frac{1}{\phi(\xi)}\left[\left(1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right)\overline{\phi''}(\xi)\right]'' = -\frac{\ddot{y}(t)}{y(t)}. \tag{8}$$

Since the first member of Eq. (8) is a function of ξ, and the second member is a function of t, both members must be equal to a constant value that is indicated as ω^2. Therefore the following two ordinary differential equations are obtained:

$$\ddot{y}(t) + \omega^2 y(t) = 0, \tag{9}$$

$$\left[\left(1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right)\overline{\phi''}(\xi)\right]'' - \omega^2\frac{mL^4}{E_0 I_0}\phi(\xi) = 0. \tag{10}$$

Eq. (10), by performing the double differentiation with respect to ξ of the first term containing Dirac's delta distribution, and after simple algebra, takes the following form:

$$\overline{\phi^{IV}}(\xi) - \alpha^4\phi(\xi) = B(\xi), \tag{11}$$

where the function $B(\xi)$ collects all the terms with Dirac's deltas and their derivatives as follows:

$$B(\xi) = \left[\sum_{i=1}^{n} \gamma_i \overline{\phi^{IV}}(\xi)\delta(\xi - \xi_{0i}) + 2\sum_{i=1}^{n} \gamma_i \overline{\phi'''}(\xi)\delta'(\xi - \xi_{0i}) + \sum_{i=1}^{n} \gamma_i \overline{\phi''}(\xi)\delta''(\xi - \xi_{0i})\right]. \tag{12}$$

In Eq. (11) the dimensionless frequency parameter $\alpha^4 = \omega^2 mL^4/E_0 I_0$ is introduced.

The solution of the first governing differential equation given by Eq. (9) is well known and guarantees that the dependence on time is harmonic. The second governing differential equation given by Eq. (11), for specified boundary conditions, leads to the evaluation of the mode shapes and the corresponding frequencies.

In order to solve Eq. (11), it can be observed that the solution $\phi(\xi)$ between two successive cracks must be of the same form of the Eigen-mode of the undamaged beam; therefore, a solution for the overall beam can be assumed as a combination of the standard trigonometric and hyperbolic functions in which the coefficients of the combination are generalised functions according to the following general form:

$$\phi(\xi) = d_1(\xi)\sin\alpha\xi + d_2(\xi)\cos\alpha\xi + d_3(\xi)\sinh\alpha\xi + d_4(\xi)\cosh\alpha\xi. \tag{13}$$

The functions $d_1(\xi)$, $d_2(\xi)$, $d_3(\xi)$, $d_4(\xi)$ appearing in Eq. (13), correspondent to the integration constants in the case of undamaged beams, are unknown generalised functions determined according to the procedure outlined in Appendix C. The expressions of $d_1(\xi)$, $d_2(\xi)$, $d_3(\xi)$, $d_4(\xi)$, dependent on four integration constants C_1, C_2, C_3, C_4, are defined below:

$$d_1(\xi) = \frac{1}{2\alpha}\sum_{i}^{n} \lambda_i M(\xi_{0i})\cos\alpha\xi_{0i} U(\xi - \xi_{0i}) + C_1,$$

$$d_2(\xi) = -\frac{1}{2\alpha}\sum_{i}^{n} \lambda_i M(\xi_{0i}) \sin \alpha\xi_{0i} U(\xi - \xi_{oi}) + C_2,$$

$$d_3(\xi) = \frac{1}{2\alpha}\sum_{i}^{n} \lambda_i M(\xi_{0i}) \cosh \alpha\xi_{0i} U(\xi - \xi_{0i}) + C_3,$$

$$d_4(\xi) = -\frac{1}{2\alpha}\sum_{i}^{n} \lambda_i M(\xi_{0i}) \sinh \alpha\xi_{0i} U(\xi - \xi_{0i}) + C_4, \tag{14}$$

where $U(\xi - \xi_{0i})$ is the well known unit step (Heaviside) function, which is the distributional derivative of Dirac's delta distribution, indicating a jump discontinuity at ξ_{0i}, $i = 1, \ldots, n$, and $M(\xi_{0i})$ are the values, evaluated at the cracked cross-sections ξ_{0i}, $i = 1, \ldots, n$, of the following normalised function:

$$M(\xi) = -\hat{M}(\xi)\frac{L^2}{E_0 I_0}, \tag{15}$$

where $\hat{M}(\xi)$ is the bending moment associated to the sought solution $\phi(\xi)$, which can be evaluated by multiplying the adopted flexural stiffness function and the second derivative of the solution, as follows:

$$\hat{M}(\xi) = -E_0 I_0 \left[1 - \sum_{i=1}^{n} \gamma_i \delta(\xi - \xi_{0i})\right] \frac{1}{L^2} \bar{\phi}''(\xi). \tag{16}$$

Moreover, in Eq. (14), the dimensionless parameters

$$\lambda_i = \frac{\gamma_i}{(1 - A\gamma_i)}, \quad i = 1, \ldots, n \tag{17}$$

are introduced and will be considered as "*damage parameters*", and adopted in the applications to represent the concentrated damages. The definition of the constant A, appearing in Eq. (17) is discussed in Appendix C. The relation between the parameters λ_i, defined in Eq. (17), and the internal rotational spring stiffnesses equivalent to the cracks, can be obtained, in view of Eq. (A.12) in Appendix A, by accounting for the spatial normalisation with respect to the length L of the beam, as follows:

$$\lambda_i = \frac{E_0 I_0}{L} \frac{1}{K_{eq,i}}, \quad i = 1, \ldots, n. \tag{18}$$

In addition, the relationship between the damage parameters λ_i and the crack depth is shown in Appendix D.

The solution $\phi(\xi)$ provided by Eq. (13), using Eq. (14), after simple algebra becomes

$$\phi(\xi) = \frac{1}{2\alpha}\sum_{i=1}^{n} \lambda_i M(\xi_{0i})[\sin \alpha(\xi - \xi_{0i}) + \sinh \alpha(\xi - \xi_{0i})]U(\xi - \xi_{0i}) + C_1 \sin \alpha\xi + C_2 \cos \alpha\xi + C_3 \sinh \alpha\xi + C_4 \cosh \alpha\xi, \tag{19}$$

where the quantities $M(\xi_{0i})$ are still unknown.

The normalised bending moment $M(\xi)$ given by Eq. (15) is a continuous function, which can be expressed, via Eq. (16) and the second distributional derivative of Eq. (19), as

$$M(\xi) = \left[1 - \sum_{i=1}^{n} \gamma_k \delta(\xi - \xi_{0k})\right] \times \left\{\frac{\alpha}{2}\sum_{k=1}^{n} \lambda_i M(\xi_{0i})[-\sin \alpha(\xi - \xi_{0i}) + \sinh \alpha(\xi - \xi_{0i})]U(\xi - \xi_{0i}) + \sum_{i=1}^{n} \lambda_i M(\xi_{0i})\delta(\xi - \xi_{0i}) - C_1\alpha^2 \sin \alpha\xi - C_2\alpha^2 \cos \alpha\xi - C_3\alpha^2 \sinh \alpha\xi - C_4\alpha^2 \cosh \alpha\xi\right\}. \tag{20}$$

The value $M(\xi_{0j})$ at the generic cracked cross-section ξ_{0j} can be selected by applying the distributional product with Dirac's delta [50–55] to the bending moment continuous function provided by Eq. (20), as

$$\begin{aligned} M(\xi_{0j}) &= \int_{-\infty}^{\infty} M(\xi)\delta(\xi - \xi_{0j})\,\mathrm{d}\xi \\ &= \frac{\alpha}{2}\sum_{i=1}^{j-1} \lambda_i M(\xi_{0i})[-\sin \alpha(\xi_{0j} - \xi_{0i}) + \sinh \alpha(\xi_{0j} - \xi_{0i})] \\ &\quad - C_1\alpha^2 \sin \alpha\xi_{0j} - C_2\alpha^2 \cos \alpha\xi_{0j} - C_3\alpha^2 \sinh \alpha\xi_{0j} - C_4\alpha^2 \cosh \alpha\xi_{0j}. \end{aligned} \tag{21}$$

It is worth noting that the expression of $M(\xi_{0j})$, provided by Eq. (21), involves the values $M(\xi_{0i})$, for $i = 1, \ldots, j-1$. The recurrence expression of the bending moments at the cracked cross-sections $M(\xi_{0i})$, for $j = 1, \ldots, n$, provided by Eq. (21), can be given by the following explicit form:

$$M(\xi_{0j}) = C_1\mu_j + C_2\upsilon_j + C_3\zeta_j + C_4\eta_j, \tag{22}$$

where

$$\mu_j = \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\mu_i[-\sin\alpha(\xi_{0j}-\xi_{0i}) + \sinh\alpha(\xi_{0j}-\xi_{0i})] - \alpha^2\sin\alpha\xi_{0j},$$

$$\upsilon_j = \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\upsilon_i[-\sin\alpha(\xi_{0j}-\xi_{0i}) + \sinh\alpha(\xi_{0j}-\xi_{0i})] - \alpha^2\cos\alpha\xi_{0j},$$

$$\zeta_j = \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\zeta_i[-\sin\alpha(\xi_{0j}-\xi_{0i}) + \sinh\alpha(\xi_{0j}-\xi_{0i})] + \alpha^2\sinh\alpha\xi_{0j},$$

$$\eta_j = \frac{\alpha}{2}\sum_{i=1}^{j-1}\lambda_i\eta_i[-\sin\alpha(\xi_{0j}-\xi_{0i}) + \sinh\alpha(\xi_{0j}-\xi_{0i})] + \alpha^2\cosh\alpha\xi_{0j}. \tag{23}$$

The solution of the Eigen-mode governing Eq. (11) is given by Eq. (19), and through Eq. (22), can be written in the following explicit form:

$$\begin{aligned}\phi(\xi) = {} & C_1\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i[\sin\alpha(\xi-\xi_{0i}) + \sinh\alpha(\xi-\xi_{0i})]U(\xi-\xi_{0i}) + \sin\alpha\xi\right\} \\ & + C_2\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\upsilon_i[\sin\alpha(\xi-\xi_{0i}) + \sinh\alpha(\xi-\xi_{0i})]U(\xi-\xi_{0i}) + \cos\alpha\xi\right\} \\ & + C_3\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i[\sin\alpha(\xi-\xi_{0i}) + \sinh\alpha(\xi-\xi_{0i})]U(\xi-\xi_{0i}) + \sinh\alpha\xi\right\} \\ & + C_4\left\{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\eta_i[\sin\alpha(\xi-\xi_{0i}) + \sinh\alpha(\xi-\xi_{0i})]U(\xi-\xi_{0i}) + \cosh\alpha\xi\right\},\end{aligned} \tag{24}$$

where the terms $\mu_i, v_i, \varsigma_i, \eta_i$ are given by Eqs. (23). The integration constants C_1, C_2, C_3, C_4 can be easily evaluated by imposing the relevant boundary conditions. The first and second derivatives of the Eigen-mode can be obtained by means of single and double differentiation of Eq. (24) by making use of the distributional derivatives of the unit step (Heaviside) function.

The parameters λ_i chosen in Eq. (24) to represent the extent of the concentrated damages allowed the formulation of the closed-form expressions for the vibration modes, regardless of the values adopted for the parameters γ_i and the constant A, which do not need to be specified.

In other words, the intensities of concentrated cracks represented by the rotational stiffness K_{eq} of equivalent springs, according to one of the models presented in the literature [31–36,56,57], are given by the parameters λ_i evaluated as in Eq. (18).

3. The frequency equation of the multi-cracked beam

The multi-cracked beam frequency equation is derived by simply enforcing the standard boundary conditions; this method even works for the general case of rotational and translational spring supports. In this section the closed-form solution presented in Eq. (24) is used for cases of simply supported (pinned–pinned, PP), cantilever (clamped–free, CF), clamped–clamped (CC), and free–free (FF) Euler–Bernoulli beam. The frequency equations are derived and numerically solved in order to obtain the frequencies of the considered multi-cracked beams, and the corresponding explicit expressions of the vibration modes. Additionally, a parametric study for different numbers, positions, and values of the damage parameters is presented.

The damage parameters λ_i were chosen to be representative of the damage intensities, and the correspondent crack depth can be easily inferred through existing damage models as reported in Appendix D.

3.1. Simply supported beam

The boundary conditions of the simply supported beam can be expressed as follows:

$$\phi(0)=0,\quad \phi''(0)=0,\quad \phi(1)=0,\quad \phi''(1)=0. \tag{25}$$

Accounting for Eqs. (24) and (25), the following conditions for the integration constants C_1, C_2, C_3, C_4, are written as

$$C_2=C_4=0, \tag{26}$$

$$\begin{bmatrix} \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i S_i(\alpha,1)+\sin\alpha & \frac{1}{2\alpha}\sum_{j=1}^{n}\lambda_j\zeta_j S_j(\alpha,1)+\sinh\alpha \\ \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i S_i'(\alpha,1)-\alpha\sin\alpha & \frac{1}{2\alpha}\sum_{j=1}^{n}\lambda_j\zeta_j S_j'(\alpha,1)+\alpha\sinh\alpha \end{bmatrix}\begin{bmatrix} C_1 \\ C_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \tag{27}$$

where for simplicity the operator $S_i(\alpha,\xi)$ has been introduced as follows:

$$S_i(\alpha,\xi)=[\sin\alpha(\xi-\xi_{0i})+\sinh\alpha(\xi-\xi_{0i})]. \tag{28}$$

The frequency equation of the simply supported multi-cracked beam is obtained by evaluating the second-order determinant of the system of Eq. (27) as follows:

$$\begin{aligned}\sin\alpha\sinh\alpha&+\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i\sin\alpha(1-\xi_{0i})\sinh\alpha+\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i\sinh\alpha(1-\xi_{0i})\sin\alpha\\ &-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\mu_i\lambda_j\zeta_i\sinh\alpha(1-\xi_{0i})\sin\alpha(1-\xi_{0j})=0\end{aligned} \tag{29}$$

the zeros of this equation indicate the values of the frequency parameters α_k. If all the damage parameters λ_i are zero, i.e. no crack occurs, Eq. (29) reduces to the frequency equation of the undamaged simply supported beam.

By substituting the frequency parameter in the boundary condition system of Eq. (27), the value of the integration constants that provide the vibration mode of the simply supported multi-cracked beam can be obtained as follows:

$$C_3=1,\quad C_1=-\frac{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\zeta_i S_i(\alpha_k,1)+\sinh\alpha_k}{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i\mu_i S_i(\alpha_k,1)+\sin\alpha_k}. \tag{30}$$

By replacing into Eq. (24) the values of the integration constants given by Eqs. (26) and (30), the closed form expressions of the vibration modes of the simply supported multi-cracked beam are obtained as follows:

$$\begin{aligned}\phi_k(\xi)=&-\frac{\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\zeta_i S_i(\alpha_k,1)+\sinh\alpha_k}{\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\mu_i S_i(\alpha_k,1)+\sin\alpha_k}\left\{\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\mu_i S_i(\alpha_k,\xi)U(\xi-\xi_{0i})+\sin\alpha_k\xi\right\}\\ &+\frac{1}{2\alpha_k}\sum_{i=1}^{n}\lambda_i\zeta_i S_i(\alpha_k,\xi)U(\xi-\xi_{0i})+\sinh\alpha_k\xi.\end{aligned} \tag{31}$$

3.2. Cantilever beam

The boundary conditions of the cantilever beam can be expressed as follows:

$$\phi(0)=0,\quad \phi'(0)=0,\quad \phi''(1)=0,\quad \phi'''(1)=0, \tag{32}$$

Accounting for Eqs. (32) and (24) leads to the following conditions for the integration constants C_1, C_2, C_3, C_4:

$$C_4=-C_2,\quad C_3=-C_1, \tag{33}$$

$$\begin{bmatrix} \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i-\zeta_i)S_i'(\alpha,1)-(\sin\alpha+\sinh\alpha) & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i-\eta_i)S_i'(\alpha,1)-(\cos\alpha+\cosh\alpha) \\ \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i-\zeta_i)S_i''(\alpha,1)-\alpha^2(\cos\alpha+\cosh\alpha) & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i-\eta_i)S_i''(\alpha,1)+\alpha^2(\sin\alpha-\sinh\alpha) \end{bmatrix}\begin{bmatrix} C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}, \tag{34}$$

where $S_i'(\alpha,1)$ and $S_i''(\alpha,1)$ are the first and second derivative, respectively, evaluated at $\xi=1$, of the $S_i(\alpha,\xi)$ function defined in Eq. (28). The evaluation of the determinant of Eq. (34) yields the frequency equation, the zeros of which denote the frequency parameters. Substituting the frequency parameters in the boundary condition system of Eq. (34) results in the integration constants that provide the vibration modes of the cantilever multi-cracked beam.

3.3. Clamped–clamped beam

For a clamped–clamped beam the following boundary conditions must be enforced

$$\phi(0)=0,\quad \phi'(0)=0,\quad \phi(1)=0,\quad \phi'(1)=0. \tag{35}$$

Using Eq. (24), the conditions for the integration constants C_1, C_2, C_3, C_4 given by Eqs. (35), can be written as

$$C_3=-C_1,\quad C_4=-C_2, \tag{36}$$

$$\begin{bmatrix} \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i-\zeta_i)S_i(\alpha,1)+\sin\alpha-\sinh\alpha & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i-\eta_i)S_i(\alpha,1)+\cos\alpha-\cosh\alpha \\ \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i-\zeta_i)S_i'(\alpha,1)+\alpha(\cos\alpha-\cosh\alpha) & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i-\eta_i)S_i'(\alpha,1)-\alpha(\sin\alpha-\sinh\alpha) \end{bmatrix}\begin{bmatrix} C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{37}$$

The zero of the second-order determinant gives the frequency parameters of the clamped–clamped beam, and after substitution into the system of Eq. (37), the corresponding eigenvector can be expressed in closed form.

3.4. Free–free beam

For a free–free beam, the boundary conditions at the left and right ends may be written as

$$\phi''(0)=0,\quad \phi'''(0)=0,\quad \phi''(1)=0,\quad \phi'''(1)=0. \tag{38}$$

The conditions (38), by making use of the closed-form expression given by Eq. (24), lead to the following expressions for the integration constants:

$$C_3=C_1,\quad C_4=C_2, \tag{39}$$

$$\begin{bmatrix} \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i+\zeta_i)S_i'(1-\xi_{0i})-\alpha(\sin\alpha+\sinh\alpha) & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i+\eta_i)S_i'(1-\xi_{0i})-\alpha(\cos\alpha+\cosh\alpha) \\ \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i+\zeta_i)S_i''(1-\xi_{0i})-\alpha^2(\cos\alpha+\cosh\alpha) & \frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i+\eta_i)S_i''(1-\xi_{0i})+\alpha^2(\sin\alpha+\sinh\alpha) \end{bmatrix}\begin{bmatrix} C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}. \tag{40}$$

By setting the system matrix determinant of Eq. (40) equal to zero, the expression of the exact frequency equation can be obtained.

Once the frequency parameter has been obtained, the solution of the boundary condition system (39) and (40) gives the value of the integration constants that provide the vibration modes of the free–free multi-cracked beam, which are written as

$$C_1=\vartheta_k C_2,\quad C_3=C_1,\quad C_4=C_2, \tag{41}$$

where

$$\vartheta_k=-\frac{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\upsilon_i+\eta_i)[-\sin\alpha_k(1-\xi_{0i})+\sinh\alpha_k(1-\xi_{0i})]-\cos\alpha_k+\cosh\alpha_k}{\frac{1}{2\alpha}\sum_{i=1}^{n}\lambda_i(\mu_i+\zeta_i)\{[-\sin\alpha_k(1-\xi_{0i})+\sinh\alpha_k(1-\xi_{0i})]-\sin\alpha_k+\sinh\alpha_k}. \tag{42}$$

By substituting the values of the integration constants given by Eqs. (41) and (42) into Eq. (24), the closed-form expressions of the vibration modes of the free–free beam with n concentrated cracks can be easily obtained.

4. Numerical applications

The applications reported in this section are for the above considered damaged beams for different numbers of concentrated cracks characterised by different positions and intensities.

Fig. 1 displays the frequency parameter of beams for different numbers of equally spaced identical concentrated damages as a function of the damage intensity parameter λ. The cases corresponding to the values $n=1,3,5,10,50$ have been considered in a range of the damage intensity parameter between zero and 10. The frequency parameter decreases rapidly at lower values of the damage parameters, which more realistically models moderate damages.

The effect of the position of a single crack in the beams is investigated in Fig. 2a, where the frequency parameter is reported as a function of the crack position ξ_0 for the values of the damage parameter $\lambda=2$, correspondent to a significant damage. The simply supported and free–free beams exhibit similar behaviours; that is when the crack moves from the

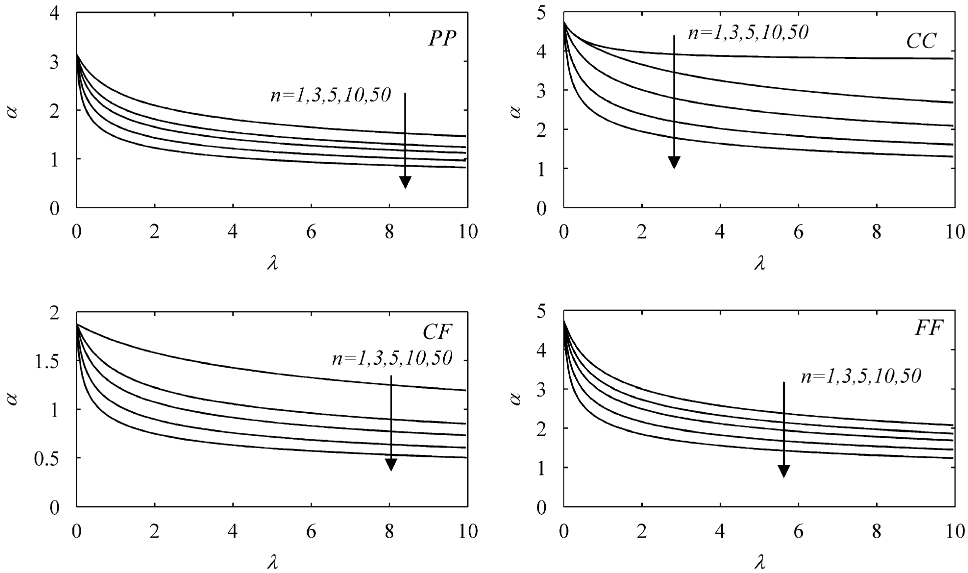

Fig. 1. First frequency parameters of multi-cracked beams for different numbers n of equally spaced concentrated damages ($n = 1, 3, 5, 10, 50$) versus the damage parameters $\lambda_i = \lambda, \quad i = 1, \ldots, n$. *PP* (pinned–pinned); *CC* (clamped–clamped); *CF* (clamped–free); *FF* (free–free).

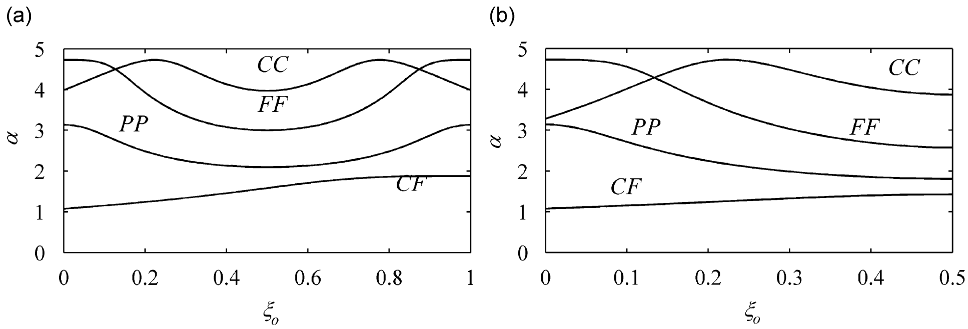

Fig. 2. First frequency parameter versus the crack positions for the value of the damage parameters $\lambda_i = 2$: (a) single cracked beams; (b) beams with two symmetric cracks. *PP* (pinned–pinned); *CC* (clamped–clamped); *CF* (clamped–free); *FF* (free–free).

boundary ends towards the middle cross-section, the frequency parameter decreases. The clamped–clamped beam responds differently, as the frequency parameter increases when the crack moves from the clamped ends towards the cross-section where the inflection point of the undamaged beam vibration mode is encountered, reaching the frequency of the undamaged beam. The frequency then decreases when the crack moves towards the middle cross-section. As expected, the cantilever beam frequency parameter increases when the crack moves from the clamped towards the free end.

The effect of the position of two cracks equidistant ξ_0 from the boundary ends of the beams and characterised by the same damage index, $\lambda = 2$, is investigated in Fig. 2b, where the frequency parameter is reported as a function of the crack distance ξ_0 from the boundary end. For each considered case, the curves appear to be similar to the case concerning a single moving crack. The frequency parameter decreases as the two cracks simultaneously approach the middle of the beam in the *PP* and *FF* cases. Analogous to the single crack case, the *CC* beam curve assumes the value of the undamaged beam when the cracks are both collocated at the inflection points of the vibration mode of the beam without cracks. The frequency parameter of the *CF* beam continuously increases as the cracks move simultaneously towards the middle cross-section of the beam.

The vibration mode variability of the previously considered beams subject to two moving cracks as a function of the crack positions ξ_0 are reported in Fig. 3 as 3D-graphs. It can be observed that the crack on the free side of the cantilever beam has negligible impact on its mode shape.

Finally, Fig. 4 shows the fifth vibration mode shape for the considered boundary conditions, which are plotted for a beam subject to 10 cracks of equally high intensity $\lambda = 10$.

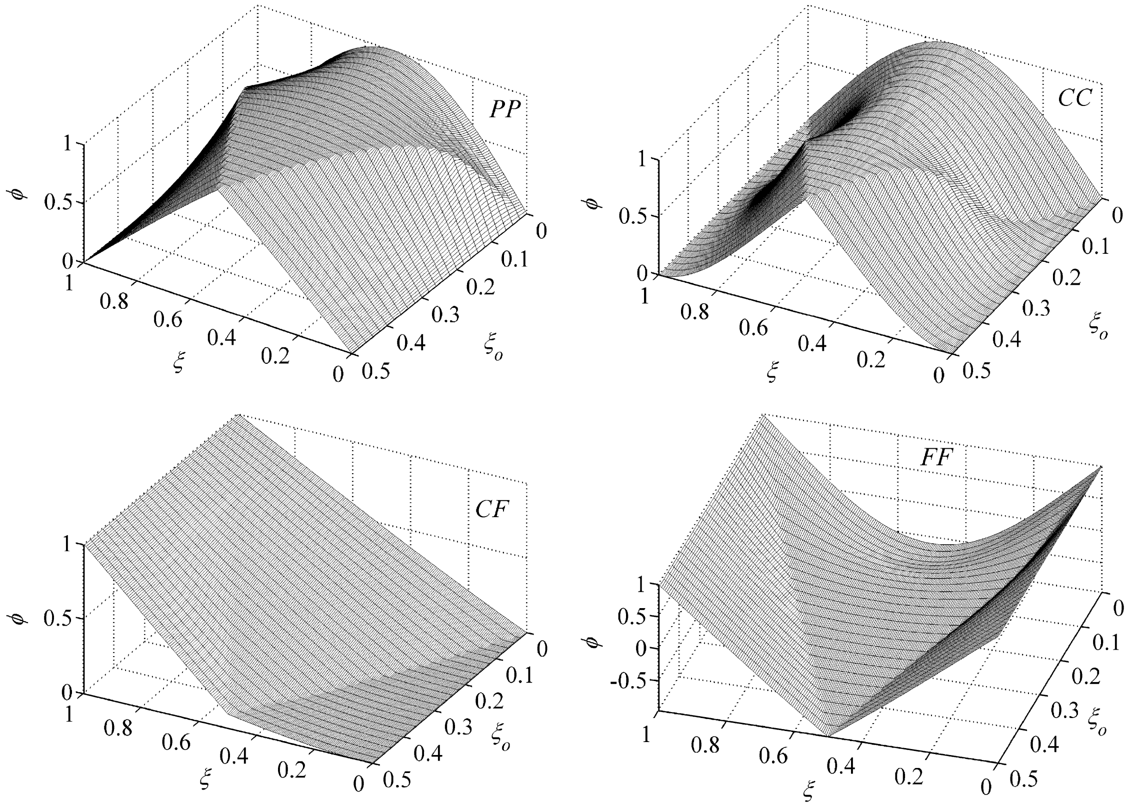

Fig. 3. Vibration mode variability in the presence of two symmetric moving cracks for the value of the damage parameters $\lambda_i = 2$. *PP* (pinned–pinned); *CC* (clamped–clamped); *CF* (clamped–free); *FF* (free–free).

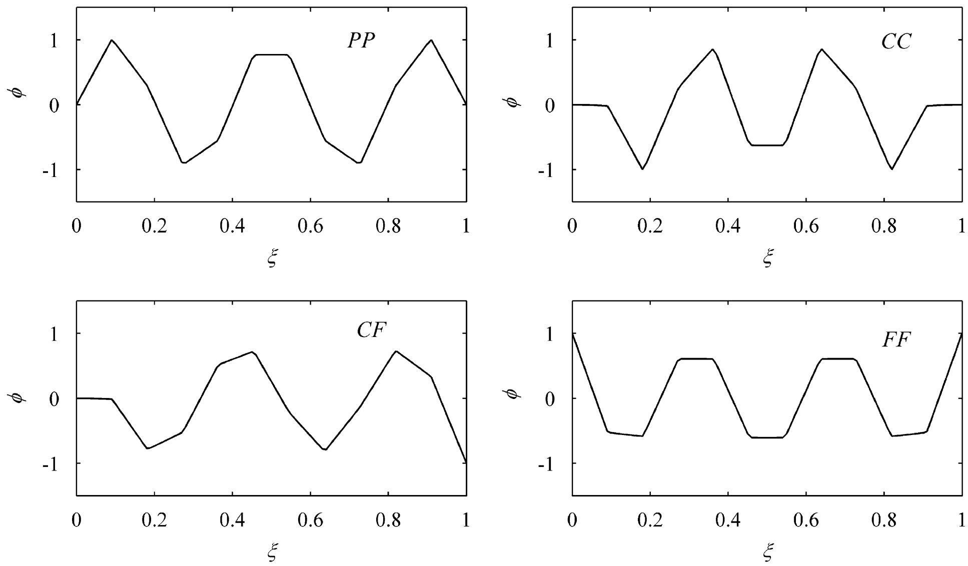

Fig. 4. Fifth vibration mode in the presence of 10 cracks for the value of the damage parameters $\lambda_i = \lambda = 10$. *PP* (pinned–pinned); *CC* (clamped–clamped); *CF* (clamped–free); *FF* (free–free).

5. Conclusions

In this paper, the exact closed form expressions for the vibration modes of the uniform Euler–Bernoulli beams in the presence of multiple concentrated cracks, modelled using Dirac's deltas, were derived incorporating the classical analytical structure of the undamaged beams. The vibration mode shapes are explicitly dependent on the concentrated cracks present along the beam, and were defined as functions of only four integration constants, which are determined by the standard boundary conditions. The presented solution allows the frequency equation for any number, position, and intensity of the cracks to be derived for any beam by enforcing the boundary conditions at the ends of the beam without additional continuity conditions. A numerical study of multi-cracked Euler–Bernoulli beams subjected to different boundary constraints, for different numbers, positions, and damage intensities was developed and discussed.

The analytical structure proposed to express the vibration mode is fundamental for the evaluation of the dynamic stiffness matrix of multi-cracked framed structures, which is a subject of a forthcoming paper devoted to performing exact modal analyses of damaged framed structures.

Moreover, the exact explicit expression of the vibration modes formulated for the case of open cracks in this work are very promising if employed in the evaluation of the nonlinear response of closing cracks; in fact, in this case the nonlinear step-by-step analysis is the focus of current research conducted by the authors, and is performed by means of a sequence of linear analyses (piecewise linear analysis).

Furthermore, the proposed integration procedure can also be extended to the case of Timoshenko beams in order to obtain closed-form expressions of the relevant vibration modes. Finally, the model adopted in this study, once extended to the case of shear-deformable beams, will allow the inclusion of discontinuities in the transversal displacement function.

Appendix A. Equivalence between the proposed model for Euler–Bernoulli beams in the presence of concentrated cracks and the standard governing equations

For convenience, the equivalence between the proposed model adopted in this work, and the standard governing equations is shown in the static case and for a single crack.

The standard governing equations of a beam with variable bending stiffness $E(x)I(x)$ having a concentrated crack at abscissa x_0, represented by the equivalent rotational stiffness K_{eq}, under a transversal static load $q(x)$, are given as follows:

$$[E(x)I(x)u''(x)]'' = q(x) \quad \text{in } (0, x_0) \cup (x_0, L), \tag{A.1}$$

$$\Delta u(x_0) = \Delta u''(x_0) = \Delta u'''(x_0) = 0, \tag{A.2}$$

$$K_{eq}\Delta u'(x_0) = E(x_0)I(x_0)u''(x_0), \tag{A.3}$$

where $u(x)$ is the transversal displacement function, the apex indicates the standard derivative with respect to the spatial variable x, and Δ indicates a jump discontinuity of the subsequent function.

The solution $u(x)$ of the governing Eqs. (A.1)–(A.3) is expected to be a continuous function with discontinuous derivative at x_0 as imposed by Eq. (A.3); hence, to address this type of function, the concept of a distributional derivative, with respect to the spatial variable x, will now be considered and is indicated with a superimposed bar.

The distributional derivative $\overline{u'}(x)$ of $u(x)$ having a discontinuity can be obtained as follows:

$$\overline{u'}(x) = u'(x) + \Delta u(x_0)\delta(x - x_0), \tag{A.4}$$

where $u'(x)$ is the standard derivative of $u(x)$ in $(0, x_0) \cup (x_0, L)$ and $\delta(x{-}x_0)$ is the well known Dirac's delta distribution.

For the case under study, since $u(x)$ is a continuous function, $\Delta u(x_0) = 0$ and $\overline{u'}(x) = u'(x)$. However, the second derivative $\overline{u''}(x)$ is

$$\overline{u''}(x) = u''(x) + \Delta u'(x_0)\delta(x - x_0). \tag{A.5}$$

Starting from Eq. (A.5), an alternative formulation for the governing equation of the Euler–Bernoulli beam that includes the cracked cross-section can be obtained. This is achieved by introducing the definition of the product of two Dirac's deltas as defined by Bagarello [57,58]: $\delta(x - x_0)\delta(x - x_0) = A\delta(x - x_0)$. In fact, if we multiply both parts of Eq. (A.5) by a function containing a Dirac's delta such as $\varphi(x)[1 - \hat{\gamma}\delta(x - x_0)]$ with $\varphi(x)$ regular function, we obtain

$$\begin{aligned}\varphi(x)[1 - \hat{\gamma}\delta(x - x_0)]\overline{u''}(x) &= \varphi(x)[1 - \hat{\gamma}\delta(x - x_0)][u''(x) + \Delta u'(x_0)\delta(x - x_0)] \\ &= \varphi(x)u''(x)[1 - \hat{\gamma}\delta(x - x_0)] + \varphi(x)\Delta u'(x_0)[1 - \hat{\gamma}A]\delta(x - x_0).\end{aligned} \tag{A.6}$$

Eq. (A.6), in view of Eq. (A.3), leads to

$$\begin{aligned}\varphi(x)[1 - \hat{\gamma}\delta(x - x_0)]\overline{u''}(x) &= \varphi(x)u''(x)[1 - \hat{\gamma}\delta(x - x_0)] + \varphi(x)\frac{E(x_0)I(x_0)}{K_{eq}}u''(x_0)[1 - \hat{\gamma}A]\delta(x - x_0) \\ &= \varphi(x)u''(x) + \varphi(x_0)u''(x_0)\left[-\hat{\gamma} + \frac{E(x_0)I(x_0)}{K_{eq}}(1 - \hat{\gamma}A)\right]\delta(x - x_0).\end{aligned} \tag{A.7}$$

Now, if we choose the parameter $\hat{\gamma}$ such that

$$\hat{\gamma} = \frac{E(x_0)I(x_0)}{K_{eq} + AE(x_0)I(x_0)}, \tag{A.8}$$

then Eq. (A.7) can be rewritten as

$$\varphi(x)[1-\hat{\gamma}\delta(x-x_0)]\overline{u''}(x)=\varphi(x)u''(x). \tag{A.9}$$

Finally, by assuming that $\varphi(x)$ is coincident with the bending stiffness $E(x)I(x)$, and utilising double standard differentiation of Eq. (A.9), we obtain

$$[E(x)I(x)(1-\hat{\gamma}\delta(x-x_0))\overline{u''}(x)]''=[E(x)I(x)u''(x)]''. \tag{A.10}$$

Eq. (A.10), in view of Eq. (A.1), yields

$$[E(x)I(x)(1-\hat{\gamma}\delta(x-x_0))\overline{u''}(x)]''=q(x). \tag{A.11}$$

The model in Eq. (A.11) can be adopted to treat the case of concentrated cracks. In fact, by incorporating Eq. (A.8), the stiffness of the rotational spring equivalent to the crack is as follows:

$$K_{\mathrm{eq}}=\frac{1-\hat{\gamma}A}{\hat{\gamma}}E(x_0)I(x_0), \tag{A.12}$$

where $\hat{\gamma}$ and A are dimensional parameters (as length and length^{-1}, respectively).

The governing equation under the form presented in Eq. (A.11) is equivalent to the standard form reported in Eqs. (A.1)–(A.3).

In addition, to show that the proposed model retains the properties of the classical formulation, the evaluation of the strain energy is pursued in Appendix B, where it is proved that when the external load induces a zero bending moment at the cracked cross-section, the solution of the undamaged beam is recovered.

The beam model with a single concentrated crack in Eq. (A.11) has been extended to the case of multiple cracks and to different types of discontinuities in [47], where closed-form solutions have been obtained in a static context.

The present work provides an extension to the dynamic field and for multiple cracks of the model presented in Eq. (A.11), which is for the case of an internal hinge endowed with a rotational spring representative of a concentrated crack in the static context.

Appendix B. The strain energy of a beam having a concentrated crack

According to the model presented in Eq. (A.11), the flexural stiffness of the Euler–Bernoulli beam in the presence of a single crack is written as

$$\overline{E(x)I(x)}=E(x)I(x)[1-\hat{\gamma}\delta(x-x_0)], \tag{B.1}$$

where $E(x)I(x)$ is a regular function representing the flexural stiffness in $(0,x_0)\cup(x_0,L)$.

The evaluation of the strain energy of the cracked beam can be performed as follows:

$$\Pi=\frac{1}{2}\int_0^L\bar{\chi}(x)\overline{E(x)I(x)}\,\bar{\chi}(x)\,\mathrm{d}x, \tag{B.2}$$

where $\bar{\chi}(x)=-\overline{u''}(x)$ is the curvature function evaluated as the second distributional derivative of the transversal displacement $u(x)$, evaluated using Eq. (A.5).

Substitution of Eqs. (A.5) and (B.1) into Eq. (B.2) gives

$$\Pi=\frac{1}{2}\int_0^L[u''(x)+\Delta u'(x_0)\delta(x-x_0)]E(x)I(x)[1-\hat{\gamma}\delta(x-x_0)][u''(x)+\Delta u'(x_0)\delta(x-x_0)]\,\mathrm{d}x. \tag{B.3}$$

The expression of the discontinuity of the rotation function given by Eq. (A.3) is utilised so that Eq. (B.3) can be written as follows:

$$\begin{aligned}\Pi=\frac{1}{2}\int_0^L&\left[u''(x)+\frac{E(x_0)I(x_0)}{K_{\mathrm{eq}}}u''(x_0)\delta(x-x_0)\right]E(x)I(x)[1-\hat{\gamma}\delta(x-x_0)]\\&\times\left[u''(x)+\frac{E(x_0)I(x_0)}{K_{\mathrm{eq}}}u''(x_0)\delta(x-x_0)\right].\end{aligned} \tag{B.4}$$

The adoption of the definition of the product of two Dirac's deltas provided in [58,59] leads to

$$\begin{aligned}\Pi=\frac{1}{2}\int_0^L&\left[u''(x)+\frac{E(x_0)I(x_0)}{K_{\mathrm{eq}}}u''(x_0)\delta(x-x_0)\right]\\&\times E(x)I(x)\left\{u''(x)+u''(x_o)\left[-\hat{\gamma}+\frac{E(x_0)I(x_0)}{K_{\mathrm{eq}}}-\hat{\gamma}\frac{E(x_0)I(x_0)}{K_{\mathrm{eq}}}A\right]\delta(x-x_0)\right\}\mathrm{d}x.\end{aligned} \tag{B.5}$$

The choice of $\hat{\gamma}$ given by Eq. (A.8) means that the strain energy Π can be given by the following form:

$$\begin{aligned}\Pi &= \frac{1}{2}\int_0^L \left[u''(x) + \frac{E(x_0)I(x_0)}{K_{\text{eq}}}u''(x_0)\delta(x-x_0)\right]E(x)I(x)u''(x)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{x_0} u''(x)E(x)I(x)u''(x)\,\mathrm{d}x + \frac{1}{2}\int_{x_0}^{L} u''(x)E(x)I(x)u''(x)\,\mathrm{d}x + \frac{1}{2}E(x_0)I(x_0)\frac{u''(x_0)}{K_{\text{eq}}}E(x_0)I(x_0)u''(x_0)\end{aligned} \tag{B.6}$$

which is the correct strain energy of the classical model of a cracked beam where the third term at the right side indicates the contribution of the internal spring equivalent to the damage.

Furthermore, Eq. (B.6) shows that when an external load induces a zero curvature at x_0 (i.e. zero bending moment at the cracked cross-section), the solution of the undamaged beam is easily recovered.

Appendix C. The integration procedure

The appendix presents a procedure to determine the functions $d_1(\xi), d_2(\xi), d_3(\xi), d_4(\xi)$, which appear in Eq. (13), and are used to define the integral of the Eigen-mode governing Eq. (11).

Eq. (13) is given here for convenience:

$$\phi(\xi) = d_1(\xi)\sin\alpha\xi + d_2(\xi)\cos\alpha\xi + d_3(\xi)\sinh\alpha\xi + d_4(\xi)\cosh\alpha\xi. \tag{C.1}$$

The $d_1(\xi), d_2(\xi), d_3(\xi), d_4(\xi)$ functions, multiplying the basic trigonometric and transcendental functions, are unknown generalised functions whose evaluation requires four independent conditions. The first condition is derived by satisfying the governing differential equation (11), which is given below for clarity:

$$\overline{\phi^{\prime v}}(\xi) - \alpha^4\phi(\xi) = B(\xi), \tag{C.2}$$

where

$$B(\xi) = \left[\sum_{i=1}^{n}\gamma_i\overline{\phi^{\prime v}}(\xi)\delta(\xi-\xi_{0i}) + 2\sum_{i=1}^{n}\gamma_i\overline{\phi'''}(\xi)\delta'(\xi-\xi_{0i}) + \sum_{i=1}^{n}\gamma_i\overline{\phi''}(\xi)\delta''(\xi-\xi_{0i})\right]. \tag{C.3}$$

Eq. (C.2) will be proposed later under an explicit form as functions of only the first derivatives of $d_1(\xi), d_2(\xi), d_3(\xi), d_4(\xi)$.

The remaining conditions can be obtained by requiring that the fourth derivative of the Eigen-mode appearing in Eq. (C.2) involves only the first distributional derivatives of the unknown functions $d_1(\xi), d_2(\xi), d_3(\xi), d_4(\xi)$, as outlined in what follows.

Distributional differentiation of Eq. (C.1) leads to

$$\begin{aligned}\overline{\phi'}(\xi) &= d_1(\xi)\alpha\cos\alpha\xi - d_2(\xi)\alpha\sin\alpha\xi + d_3(\xi)\alpha\cosh\alpha\xi + d_4(\xi)\alpha\sinh\alpha\xi \\ &\quad + \overline{d'}_1(\xi)\sin\alpha\xi + \overline{d'}_2(\xi)\cos\alpha\xi + \overline{d'}_3(\xi)\sinh\alpha\xi + \overline{d'}_4(\xi)\cosh\alpha\xi.\end{aligned} \tag{C.4}$$

By imposing the condition that

$$\overline{d'}_1(\xi)\sin\alpha\xi + \overline{d'}_2(\xi)\cos\alpha\xi + \overline{d'}_3(\xi)\sinh\alpha\xi + \overline{d'}_4(\xi)\cosh\alpha\xi = 0 \tag{C.5}$$

the second distributional derivative of $\phi(\xi)$ is obtained as

$$\begin{aligned}\overline{\phi''}(\xi) &= -d_1(\xi)\alpha^2\sin\alpha\xi - d_2(\xi)\alpha^2\cos\alpha\xi + d_3(\xi)\alpha^2\sinh\alpha\xi + d_4(\xi)\alpha^2\cosh\alpha\xi \\ &\quad + \overline{d'}_1(\xi)\alpha\cos\alpha\xi - \overline{d'}_2(\xi)\alpha\sin\alpha\xi + \overline{d'}_3(\xi)\alpha\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha\sinh\alpha\xi.\end{aligned} \tag{C.6}$$

Furthermore, by imposing the condition that

$$\overline{d'}_1(\xi)\alpha\cos\alpha\xi - \overline{d'}_2(\xi)\alpha\sin\alpha\xi + \overline{d'}_3(\xi)\alpha\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha\sinh\alpha\xi = 0 \tag{C.7}$$

the third distributional derivative of $\phi(\xi)$ is obtained as

$$\begin{aligned}\overline{\phi'''}(\xi) &= -d_1(\xi)\alpha^3\cos\alpha\xi + d_2(\xi)\alpha^3\sin\alpha\xi + d_3(\xi)\alpha^3\cosh\alpha\xi + d_4(\xi)\alpha^3\sinh\alpha\xi \\ &\quad - \overline{d'}_1(\xi)\alpha^2\sin\alpha\xi - \overline{d'}_2(\xi)\alpha^2\cos\alpha\xi + \overline{d'}_3(\xi)\alpha^2\sinh\alpha\xi + \overline{d'}_4(\xi)\alpha^2\cosh\alpha\xi.\end{aligned} \tag{C.8}$$

Finally, by imposing the additional condition that

$$-\overline{d'}_1(\xi)\alpha^2\sin\alpha\xi - \overline{d'}_2(\xi)\alpha^2\cos\alpha\xi + \overline{d'}_3(\xi)\alpha^2\sinh\alpha\xi + \overline{d'}_4(\xi)\alpha^2\cosh\alpha\xi \tag{C.9}$$

the fourth distributional derivative of $\phi(\xi)$ is obtained as

$$\begin{aligned}\overline{\phi^{\prime v}}(\xi) &= d_1(\xi)\alpha^4\sin\alpha\xi + d_2(\xi)\alpha^4\cos\alpha\xi + d_3(\xi)\alpha^4\sinh\alpha\xi + d_4(\xi)\alpha^4\cosh\alpha\xi \\ &\quad - \overline{d'}_1(\xi)\alpha^3\cos\alpha\xi + \overline{d'}_2(\xi)\alpha^3\sin\alpha\xi + \overline{d'}_3(\xi)\alpha^3\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha^3\sinh\alpha\xi.\end{aligned} \tag{C.10}$$

The governing Eq. (C.2), to be fulfilled by the desired solution, can be obtained under an explicit form involving only the first distributional derivatives of the $d_1(\xi), d_2(\xi), d_3(\xi), d_4(\xi)$ functions by making use of Eqs. (C.1) and (C.10) as follows:

$$-\overline{d'}_1(\xi)\alpha^3\cos\alpha\xi + \overline{d'}_2(\xi)\alpha^3\sin\alpha\xi + \overline{d'}_3(\xi)\alpha^3\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha^3\sinh\alpha\xi = B(\xi). \tag{C.11}$$

Incorporating the assumed conditions given by Eqs. (C.5), (C.7), (C.9), and (C.11) the generalised functions $d_1(\xi), d_2(\xi), d_3(\xi)$ and $d_4(\xi)$ can be obtained by integrating the following system of four differential equations:

$$\begin{cases} \overline{d'}_1(\xi)\sin\alpha\xi + \overline{d'}_2(\xi)\cos\alpha\xi + \overline{d'}_3(\xi)\sinh\alpha\xi + \overline{d'}_4(\xi)\cosh\alpha\xi = 0, \\ \overline{d'}_1(\xi)\alpha\cos\alpha\xi - \overline{d'}_2(\xi)\alpha\sin\alpha\xi + \overline{d'}_3(\xi)\alpha\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha\sinh\alpha\xi = 0, \\ -\overline{d'}_1(\xi)\alpha^2\sin\alpha\xi - \overline{d'}_2(\xi)\alpha^2\cos\alpha\xi + \overline{d'}_3(\xi)\alpha^2\sinh\alpha\xi + \overline{d'}_4(\xi)\alpha^2\cosh\alpha\xi = 0, \\ -\overline{d'}_1(\xi)\alpha^3\cos\alpha\xi + \overline{d'}_2(\xi)\alpha^3\sin\alpha\xi + \overline{d'}_3(\xi)\alpha^3\cosh\alpha\xi + \overline{d'}_4(\xi)\alpha^3\sinh\alpha\xi = B_p(\xi). \end{cases} \tag{C.12}$$

The system of differential equations (C.12), in combination with Eq. (C.3), can be written under the following uncoupled form:

$$\overline{d'}_1(\xi) = -\frac{\cos\alpha\xi}{2\alpha^3}\left[\overline{\phi'^{v}}(\xi)\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i}) + 2\overline{\phi'''}(\xi)\sum_{i=1}^{n}\gamma_i\delta'(\xi-\xi_{0i}) + \overline{\phi''}(\xi)\sum_{i=1}^{n}\gamma_i\delta''(\xi-\xi_{0i})\right],$$

$$\overline{d'}_2(\xi) = \frac{\sin\alpha\xi}{2\alpha^3}\left[\overline{\phi'^{v}}(\xi)\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i}) + 2\overline{\phi'''}(\xi)\sum_{i=1}^{n}\gamma_i\delta'(\xi-\xi_{0i}) + \overline{\phi''}(\xi)\sum_{i=1}^{n}\gamma_i\delta''(\xi-\xi_{0i})\right],$$

$$\overline{d'}_3(\xi) = \frac{\cosh\beta\xi}{2\alpha^3}\left[\overline{\phi'^{v}}(\xi)\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i}) + 2\overline{\phi'''}(\xi)\sum_{i=1}^{n}\gamma_i\delta'(\xi-\xi_{0i}) + \overline{\phi''}(\xi)\sum_{i=1}^{n}\gamma_i\delta''(\xi-\xi_{0i})\right],$$

$$\overline{d'}_4(\xi) = -\frac{\sinh\beta\xi}{2\alpha^3}\left[\overline{\phi'^{v}}(\xi)\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i}) + 2\overline{\phi'''}(\xi)\sum_{i=1}^{n}\gamma_i\delta'(\xi-\xi_{0i}) + \overline{\phi''}(\xi)\sum_{i=1}^{n}\gamma_i\delta''(\xi-\xi_{0i})\right]. \tag{C.13}$$

Integration of Eqs. (C.13) leads to

$$d_1(\xi) = \frac{1}{2\alpha}\left[\int\overline{\phi''}(\xi)\cos\alpha\xi\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i})\right] + C_1,$$

$$d_2(\xi) = -\frac{1}{2\alpha}\left[\int\overline{\phi''}(\xi)\sin\alpha\xi\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i})\right] + C_2,$$

$$d_3(\xi) = \frac{1}{2\alpha}\left[\int\overline{\phi''}(\xi)\cosh\alpha\xi\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i})\right] + C_3,$$

$$d_4(\xi) = -\frac{1}{2\alpha}\left[\int\overline{\phi''}(\xi)\sinh\alpha\xi\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{oi})\right] + C_4, \tag{C.14}$$

where C_1, C_2, C_3, C_4 are integration constants. The functions $d_1(\xi)$, $d_2(\xi)$, $d_3(\xi)$, and $d_4(\xi)$ provided by Eqs. (C.14) depend on integrals of the function $\overline{\phi''}(\xi)$ multiplied by continuous functions, and by Dirac's delta generalised functions.

To evaluate these integrals, it is important to note that the function $\overline{\phi''}(\xi)$ appearing in Eqs. (C.14), under the flexural stiffness model adopted in this work, is a singular function that is related to the continuous function of the flexural moment $\hat{M}(\xi)$, and is associated with the solution via the following expression:

$$\hat{M}(\xi) = -E_0I_0\left[1-\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i})\right]\overline{\phi''}(\xi)\frac{1}{L^2}. \tag{C.15}$$

The continuous function of the flexural moment $\hat{M}(\xi)$ can be normalised for convenience:

$$M(\xi) = -\hat{M}(\xi)\frac{L^2}{E_0I_0} = \left[1-\sum_{i=1}^{n}\gamma_i\delta(\xi-\xi_{0i})\right]\overline{\phi''}(\xi). \tag{C.16}$$

Multiplying both parts of Eq. (C.16) by $f(\xi)\delta(\xi-\xi_{0j})$, with $f(\xi)$ as a continuous function, followed by integration and some algebra, leads to

$$\int\overline{\phi''}(\xi)f(\xi)\gamma_j\delta(\xi-\xi_{0j})d\xi = M(\xi_{0j})f(\xi_{0j})\lambda_jU(\xi-\xi_{0j}), \tag{C.17}$$

in which the following definition of the product of two Dirac's deltas have been used [58]:

$$\delta(\xi-\xi_{0j})\delta(\xi-\xi_{0k}) = \begin{cases} A\delta(\xi-\xi_{0j}), & j=k, \\ 0, & j\neq k, \end{cases} \tag{C.18}$$

where a set of values for the quantity A, for which Eq. (C.18) holds, is defined by Bagarello [59].

In Eq. (C.17) the following dimensionless parameters:

$$\lambda_i = \frac{\gamma_i}{(1-A\gamma_i)}, \quad i=1,\ldots,n \tag{C.19}$$

have been introduced and will be considered as "*damage parameters*" and adopted in the applications to represent the concentrated damages.

The functions $d_1(\xi)$, $d_2(\xi)$, $d_3(\xi)$, $d_4(\xi)$, provided by Eqs. (C.14) and using Eq. (C.17), can now be written as follows:

$$d_1(\xi) = \frac{1}{2\alpha}\sum_i^n \lambda_i M(\xi_{0i})\cos\alpha\xi_{0i}U(\xi-\xi_{0i}) + C_1,$$

$$d_2(\xi) = -\frac{1}{2\alpha}\sum_i^n \lambda_i M(\xi_{0i})\sin\alpha\xi_{0i}U(\xi-\xi_{0i}) + C_2,$$

$$d_3(\xi) = \frac{1}{2\alpha}\sum_i^n \lambda_i M(\xi_{0i})\cosh\alpha\xi_{0i}U(\xi-\xi_{0i}) + C_3,$$

$$d_4(\xi) = -\frac{1}{2\alpha}\sum_i^n \lambda_i M(\xi_{0i})\sinh\alpha\xi_{0i}U(\xi-\xi_{0i}) + C_4. \tag{C.20}$$

Substituting Eqs. (C.20) into Eq. (C.1) provides a suitable form of the Eigen-mode to be used to obtain the explicit closed-form solution of the problem of interest.

Appendix D. Relationship between the damage parameter and the crack depth

In this appendix the damage parameters λ_i, which represent concentrated damages and related to the singularity parameters γ_i, are shown to be related to the depth of concentrated cracks by incorporating some classical crack models provided in the literature.

Eq. (18) gives the relationship between the damage parameters λ_i and the rotational spring stiffness values $K_{\mathrm{eq},i}$, which are equivalent to the concentrated damages, as follows:

$$\lambda_i = \frac{E_0 I_0}{L}\frac{1}{K_{\mathrm{eq},i}}, \quad i=1,\ldots,n. \tag{D.1}$$

It should be noted that: for $\lambda_i=0$, correspondent to the presence of no crack, Eq. (D.1) becomes $K_{\mathrm{eq},i}=\infty$; on the other hand, for $\lambda_i=\infty$, correspondent to an entirely damaged cross-section, Eq. (D.1) becomes $K_{\mathrm{eq},i}=0$.

A common approach for modelling the effect of concentrated cracks on the flexural stiffness is based on the introduction of an elastic hinge (a local compliance), which macroscopically quantifies the relation between the applied load and the strain concentration surrounding the crack [27–30]. This method gives expressions for the elastic rotational spring stiffness equivalent to the crack dependent on the crack depth.

For example, when a crack of uniform depth d is present in a rectangular cross-section of width b and height h, the following expression for the stiffness K_{eq} is used to unify the treatment of the models proposed in the literature:

$$K_{\mathrm{eq}} = \frac{E_0 I_0}{h}\frac{1}{C(\beta)}, \tag{D.2}$$

where $\beta = d/h$ is defined as the ratio between the crack depth d and the cross-section height h, and $C(\beta)$ is the dimensionless local compliance.

According to Liebowitz and Claus [31], Liebowitz et al. [32], Rizos et al. [35], and Okamura et al. [56], the local compliance $C(\beta)$ computed from the strain energy density function takes the following form:

$$\begin{aligned} C(\beta) = 5.346(&1.86\beta^2 - 3.95\beta^3 + 16.375\beta^4 - 37.226\beta^5 \\ &+ 76.81\beta^6 - 126.9\beta^7 + 172\beta^8 - 143.97\beta^9 + 66.56\beta^{10}). \end{aligned} \tag{D.3}$$

Ostachowicz and Krawczuk [33] instead proposed the following expression for the local compliance $C(\beta)$:

$$C(\beta) = 6\pi\beta^2(0.6384 - 1.035\beta + 3.7201\beta^2 - 5.1773\beta^3 + 7.553\beta^4 - 7.332\beta^5 + 2.4909\beta^6). \tag{D.4}$$

It must be noted that the crack models based on a continuous description of the beam stiffness reduction in the vicinity of the crack can be approximated by an approach that lumps flexibility by imposing that the rotation discontinuity due to the concentrated flexibility reproduces the relative rotation of the cross-sections affected by the crack.

For example, the model proposed by Bilello [36] gives the following expression for the local compliance $C(\beta)$:

$$C(\beta) = \frac{\beta(2-\beta)}{0.9(\beta-1)^2}. \tag{D.5}$$

Chondros et al. [57] provided a lumped cracked flexibility model equivalent to their continuous model using the following expression for $C(\beta)$:

$$\begin{aligned} C(\beta) = 6\pi(1-\nu^2)(0.6272\beta^2 - 1.04533\beta^3 + 4.5948\beta^4 - 9.9736\beta^5 + 20.2948\beta^6 \\ - 33.0351\beta^7 + 47.1063\beta^8 - 40.7556\beta^9 + 19.6\beta^{10}). \end{aligned} \tag{D.6}$$

The relationship between the damage parameters λ_i adopted in this work and the classical crack models can now be obtained by substituting the equivalent stiffness K_{eq}, which is given by Eq. (D.2), into Eq. (D.1), and is written for the i-th crack as follows:

$$\lambda_i = \frac{h}{L} C(\beta_i), \quad i = 1, \ldots, n. \tag{D.7}$$

Eq. (D.7) provides the relationship between the damage parameters λ_i and the dimensionless local compliance $C(\beta_i)$, given by the models reported in Eqs. (D.3)–(D.6). Furthermore, Eq. (D.7) means that the physical meaning for the damage parameters λ_i as "dimensionless local compliance", due to the cracks, and normalised with respect to the ratio of L/h of the beam, can be inferred.

The damage parameters λ_i have been directly related to the crack depth β_i, via Eq. (D.7), in order to make the present approach independent of the values of the parameters γ_i and of the constant A.

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